Interpretation of criteria weights in multicriteria decision making

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Abstract

Multicriteria decision making models are characterized by the need to evaluate a finite set of alternatives with respect to multiple criteria. The criteria weights in different aggregation rules have different interpretations and implications which have been misunderstood and neglected by many decision makers and researchers. By analyzing the aggregation rules, identifying partial values, specifying explicit measurement units and explicating direct statements of pairwise comparisons of preferences, we identify several plausible interpretations of criteria weights and their appropriate roles in different multicriteria decision making models. The underlying issues of scale validity, commensurability, criteria importance and rank consistency are examined. © 2000 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Multicriteria decision making (MCDM) models are characterized by the need to evaluate a finite set of alternatives with respect to multiple criteria. The main purpose in most MCDM problems is to measure the overall preference values of the alternatives on some permissible scale. Alternatives are generally first evaluated explicitly with respect to each of the criteria to obtain some sort of criterion specific priority scores which are then aggregated into overall preference values. These criterion specific scores and overall values may be in ordinal, interval or ratio scales. Ordinal scales on the overall preference values are sufficient if only the best alternative needs to be selected. Interval scales are used in multiattribute utility theory.
(MAUT) and most explicit multiattribute value (MAVT) function models [1,7,10,17]. Ratio scales are assumed in the analytic hierarchy process (AHP) [1,10,20,22,23].

Many different methods have been proposed for assessing criteria weights [2,5,7,13,14,24,30] which are then used explicitly to aggregate criterion specific priority scores. Thus, the true meaning and the validity of these criteria weights are crucial in order to avoid improper use of the MCDM models. Unfortunately, criteria weights are often misunderstood and misused [1,12,21,22,30], and there is no consensus on their meaning [22].

Many researchers and analysts have dismissed the difficulty in measuring and interpreting the criteria weights by merely stating that criteria weights reflect criteria importance and assume that the meaning of criteria importance is transparent and well understood by all decision makers (DMs). As a result, no definition of criteria weights is given and the users of many MCDM software packages are required to give direct answers to questions such as “Is the criterion $C_q$ of greater, minor or the same importance when compared with the criterion $C_k$?”, “How much more important is this attribute than the least important one?” and “Comparing the criteria $C_q$ and $C_k$, which one is more important and by how many times?”. Such questions are unlikely to produce correct results since criteria weights can have many different plausible and distinct interpretations. This problem will get worse with the proliferation of end-user MCDM software packages [21].

The procedures for deriving criteria weights should not be independent of the manner they are used. Although several researchers have addressed this concern and provided some partial answers [1,5,13,17,21,23,26,30], a comprehensive examination of the problem across MCDM models has not yet appeared in the literature. In this work, we identify plausible interpretations of criteria weights and their roles in different MCDM models by analyzing the aggregation rules, identifying partial values, explicating direct statements of pairwise comparisons of preferences and specifying explicit measurement units. We pay special attention to appropriate questions posed to the DM for eliciting information related to criteria weights and criteria importance. The underlying issues of commensurability, scale validity, criteria importance and consistency are examined.

2. Terminology and past research

In MCDM with $p$ criteria $C_1, \ldots, C_p$, each alternative $x$ is represented by a vector $x = (x_1, \ldots, x_p)$ where $x_k$ (for $k = 1, 2, \ldots, p$) is a raw measure or description of the tangible or intangible impact of $x$ in the criterion $C_k$ (e.g. salary of a job, color of a car). Let $S = \{x^1, \ldots, x^m\}$ denote the set of all the alternatives under evaluation. We assume that the preference of the alternatives $x^1, \ldots, x^m$ with respect to a single criterion $C_k$ is completely known and measured explicitly in an interval scale or ratio scale in which more is preferred to less. We shall denote the criterion specific score of the alternative $x^i$ with respect to the criterion $C_k$ by $z_k(x^i)$ in an interval scale and by $r_k(x^i)$ in a ratio scale. The general MCDM problem is to evaluate $Z(x^i) = [z_1(x^i), \ldots, z_p(x^i)]$ or $R(x^i) = [r_1(x^i), \ldots, r_p(x^i)]$, $i = 1, \ldots, m$ and determine the overall values of the alternatives or simply select the best alternative $x^*$. The superfunction approach [29] assumes that the overall value of $Z(x^i)$ or $R(x^i)$ can be explicitly represented by some unidimensional overall value function $U(x^i)$ so that $x^i$ is
preferred to the alternative $x^j$ if and only if $U(x^i) > U(x^j)$, and the best alternative $x^*$ has the largest $U(x^*)$ in $\{U(x^1), \ldots, U(x^m)\}$. In general, the mathematical representation of the superfunction $U$ can be complicated without any explicit notions of criteria weights. However, it is common in many MCDM models to assume the additive form $U(Z(x^i)) = \sum w_k z_k(x^i)$ or $U(R(x^i)) = \sum w_k r_k(x^i)$ where $w_k$ denotes the weight of the criterion $C_k$ [3,26]. Then, the DM needs only to determine the criteria weights $w_1, \ldots, w_p$, and the criterion specific scores in $Z(x^i)$ or $R(x^i)$ [3,5,8,13,17,26,30]. Recently, the multiplicative form

$$U(Z(x^i)) = \prod z_k(x^i)^{w_k}$$

has been proposed [4,9,12] as an alternative to AHP.

Evaluation of the alternatives under different criteria usually involves different and non-commensurate measuring scales. To truly assimilate the MCDM problem, the criteria must be compared explicitly and allowed to compete with each other [29]. It is useful to introduce intermediary information $Z(x^i)$ or $R(x^i)$ to explain the evaluation process from $x^i$ to $U(Z(x^i))$ or $U(R(x^i))$. In context-dependent preferences [27], $Z(x^i)$ are extended to $Z(x^i, S)$ to allow the criterion specific scores of $x^i$ to depend on all the alternatives in $S$.

In MAVT [28], $z_k(x^i)$ are measured on some interval scale with values generally in the interval $[0, 1]$. In AHP [20], $r_k(x^i)$ are measured on some ratio scale and $r_k(x^i)/r_k(x^j)$ is the relative ratio of the desirability of the alternative $x^i$ to the alternative $x^j$ in terms of criterion $C_k$. The contribution of $r_k(x^i)$ towards the aggregated value $U(R(x))$ is called the partial value (global weight or part-worth [11]) of $x^i$ in the criterion $C_k$ and is denoted by $T_k(x^i)$. The measurements $z_k(x^i)$, $r_k(x^i)$ and $T_k(x^i)$ are usually in different scales and non-commensurate measurement units. Showing these measurement units explicitly in the model presentation can minimize any unnecessary ambiguity and misunderstanding.

Let $\varnothing_k$, $\bar{\varnothing}_k$ and $\Theta$ denote the units of measurement for $z_k(x^i)$, $r_k(x^i)$ and the overall preference value of $x^i$, respectively. These abstract units $\varnothing_k$, $\bar{\varnothing}_k$ and $\Theta$ exist implicitly and may not be known explicitly. Then, the score of $x^i$ in terms of the criterion $C_k$ is $z_k(x^i)\varnothing_k$ or $r_k(x^i)\bar{\varnothing}_k$ for which the larger values are preferred. In the additive model $U(R(x^i))\Theta = (\sum w_k r_k(x^i))\Theta$ (respectively $U(Z(x^i))\Theta = (\sum w_k z_k(x^i))\Theta$), the measurement unit $\Theta$ represents a special common commensurate currency in which the contributions of all criteria can be measured and compared (ex poste and implicit). The criterion weight $w_k$ acts as a converting factor which is measured in $\Theta/\bar{\varnothing}_k$ or $\Theta/\varnothing_k$. A statement such as “128 units in the $k$th criterion is worth one unit in overall value” can now be stated more concisely as “128 $\bar{\varnothing}_k = 1\Theta$” and is captured in the conversion factor $w_k\Theta/\bar{\varnothing}_k = (1/128)\Theta/\bar{\varnothing}_k$. This is similar to the weights applied to the deviation variables in goal programming to express each deviation in equivalent terms. Both $\varnothing_k$ and $\Theta$ are interval scales in MAVT, and both $\bar{\varnothing}_k$ and $\Theta$ are in ratio scales in AHP.

It is difficult, if not impossible, to identify and measure $\varnothing_k$, $\bar{\varnothing}_k$ and $\Theta$ explicitly. Thus, most MCDM models employ special terms such as utilities, ratings, priorities, relative importance factors and overall values. Nonetheless, the basic rules of mathematical manipulation of $\varnothing_k$, $\bar{\varnothing}_k$ and $\Theta$ must be observed. These necessary mathematical rules would become crystal clear when $\varnothing_k$, $\bar{\varnothing}_k$ and $\Theta$ are shown explicitly in all mathematical computations.

In AHP, the linear form $U(R(x^i))\Theta = \sum w_k r_k(x^i)\Theta$ is used to compute the overall value.
The criteria weights $w_k$ are derived explicitly from pairwise comparisons matrix without using $\Theta/\alpha$ directly. It is important to note that the process of normalization does not remove units of measurement. Although the normalized priority is a proportion, the question is, a proportion of what? The local priorities are measured in $\alpha_k$ which refers to the denominator used in the normalization process. Suppose price is a criterion and the preference of DM is linear on prices in dollars. Then instead of referring to a priority of 0.26 with no units, we use $0.26/\alpha_k$ which can be interpreted as “0.26 portion of the sum $\alpha$ of the prices of all alternatives”. The linear form $U(R(x^i))\Theta = \sum w_k r_k(x^i)\Theta$ is also assumed in the additive part-worth model for conjoint analysis and preference decomposition [11] in which the partial values $T_k(x^i)\Theta = w_k r_k(x^i)\Theta$ are estimated directly without determining the criteria weights. The overall value of $x^i$ is given by the sum of partial values: $U(R(x^i))\Theta = \sum T_k(x^i)\Theta$. The direct use of the partial values eliminates the need to determine the criteria weights and is instinctively straightforward [31].

3. An illustrative example

Consider three cars $x^1$, $x^2$ and $x^3$ to be evaluated with respect to two ($p = 2$) criteria $C_1 = $ fuel efficiency in $$/km$ and $C_2 = $ size of engine in $cm^3$ with the following criteria impacts:

<table>
<thead>
<tr>
<th>Car</th>
<th>Fuel (km$^{-1}$)</th>
<th>Size (cm$^3$)</th>
<th>Alternative $x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$0.085$</td>
<td>200</td>
<td>$x^1 = (0.085, 200)$</td>
</tr>
<tr>
<td>2</td>
<td>$0.078$</td>
<td>150</td>
<td>$x^2 = (0.078, 150)$</td>
</tr>
<tr>
<td>3</td>
<td>$0.105$</td>
<td>250</td>
<td>$x^3 = (0.105, 250)$</td>
</tr>
</tbody>
</table>

Evidently, $x^2$ has the best fuel efficiency at $0.078$ km$^{-1}$ and $x^3$ has the worst fuel efficiency at $0.105$ km$^{-1}$. The ordinal ranking of engine sizes of the cars need not be monotone and may change with different DMs even though one may argue that most DMs would prefer larger engine sizes due to better power. It is conceivable that some DMs may rate $200$ cm$^3$ as the best size and $250$ cm$^3$ as the worst size for these three cars. We note that the measurement unit $\Theta$, implicitly used to measure the overall values, is intangible and not known explicitly.

1. Assume that the criteria $C_1$ and $C_2$ can be measured in $\hat{C}_1$ and $\hat{C}_2$ units which are in interval scales, and $U(Z(x^i))\Theta = (w_1 z_1(x^i) + w_2 z_2(x^i))\Theta$, $i = 1$, 2, 3. Using a direct rating method, we set $z_1(x^1)\hat{C}_1 = 1\hat{C}_1$, $z_1(x^2)\hat{C}_1 = 0\hat{C}_1$ and determine $z_1(x_1)$ subjectively (say $z_1(x_1) = 0.60\hat{C}_1$). Similarly, the ratings $z_2(x^1)\hat{C}_2$, $z_2(x^2)\hat{C}_2$ and $z_2(x^3)\hat{C}_2$ are determined by evaluating $x^1$, $x^2$ and $x^3$ under $C_2$. Suppose $z_2(x^1)\hat{C}_2 = 1\hat{C}_2$, $z_2(x^2)\hat{C}_2 = 0.7\hat{C}_2$ and $z_2(x^3)\hat{C}_2 = 0\hat{C}_2$. We note that ((1/$w_1$)$\hat{C}_1$, 0$\hat{C}_2$) and (0$\hat{C}_1$, (1/$w_2$)$\hat{C}_2$) have the same overall value of 1$\Theta$. Then, the overall values of $x^1$, $x^2$ and $x^3$ are $(0.6w_1 + w_2)\Theta$, $(w_1 + 0.7w_2)\Theta$ and $(0w_1 + w_2)\Theta$, respectively.

2. Assume that the criteria $C_1$ and $C_2$ can be measured in $\bar{R}_1$ and $\bar{R}_2$ units which are in ratio scales, and $U(R(x^i))\Theta = w_1 r_1(x^i)\Theta + w_2 r_2(x^i)\Theta$, $i = 1$, 2, 3. Using $x^1$ as a linking
alternative [23], we have $r_1(x^1)^\Theta_1 = 1^\Theta_1$, $r_2(x^1)^\Theta_2 = 1^\Theta_2$. Then the criteria weights are $w_1\Theta/\Theta_1$ and $w_2\Theta/\Theta_2$, respectively. The value $w_1/w_2$ reflects the ratio of the worth of $1^\Theta_1$ to the worth of $1^\Theta_2$. We note that the value of the linking alternative $x^1$ in the criterion $C_k$ is used as the base unit $1^\Theta_k$ which has an overall value of $w_k\Theta$. The local priorities of $x^2$ and $x^3$ are obtained by subjectively comparing $x^2$ and $x^3$ with $x^1$ under $C_1$ and $C_2$ independently. Unless the preference on fuel is linear in $$/km, we do not necessarily have $r_1(x^2) = (0.085/0.078)^\Theta_1$ and $r_1(x^3) = (0.085/0.105)^\Theta_1$. Then, the overall values of $x^1$, $x^2$ and $x^3$ are $(w_1 + w_2)\Theta$, $(w_1 r_1(x^2) + w_2 r_2(x^2))\Theta$ and $(w_1 r_1(x^3) + w_2 r_2(x^3))\Theta$, respectively.

4. Interpretation of criteria weights

Criteria weights have played an important role in measuring the overall preference values of the alternatives in many MCDM models. Based on the different assumptions on $U(Z(x))$ or $U(R(x))$, MCDM models have different aggregation rules which use the criteria weights in different ways. Distinct methods for assessing criteria weights are designed for different aggregation rules. It is of utmost importance that the DM understands the true meaning of criteria weights. Moreover, the questions posed to the DM in the elicitation process must convey the correct intended meaning. The questions posed to the DM should be direct and simple but not compromising the underlying theoretical validity. In MCDM literature, the criteria weights $w_1, \ldots, w_p$ have been given a diverse array of plausible interpretations associated with the following:

1. marginal contribution per unit of $z_k(x)$ or $r_k(x)$,
2. indifference trade-offs or rates of substitution,
3. gradient of the overall value function $U(Z(x))$ or $U(R(x))$,
4. scaling factors converting into commensurate overall value,
5. $U(Z(x))\Theta = (\sum w_k z_k(x))\Theta$ or $U(R(x))\Theta = \sum w_k r_k(x)\Theta$ is linear,
6. relative contribution of the average criterion specific scores,
7. discriminating power of the criteria on the alternatives,
8. relative contribution of swing from worse to best on each criterion,
9. vote values in binary choices,
10. relative contribution of the criteria at the optimal alternative,
11. parameters used in interactive optimization,
12. relative information content carried in the criteria,
13. relative functional importance of the criteria.

Most of the above interpretations are clear and meaningful in the immediate context of the problem. Even though some of the interpretations are very similar to each other, some of them are distinctly different and are not mutually compatible. Thus, the users of every MCDM software must be given an appropriate definition of criteria weights and criteria importance in connection with how they are used in the model. For each plausible interpretation of criteria weights, we identify below the relevant aggregation rule, the correct weight assessing method and appropriate questions posed to the DM in the elicitation process.
4.1. Weights as marginal contribution

The overall value function \( U(Z(x)) \) maps \( z = [z_1(x), \ldots, z_p(x)] \) into \( U(z) \Theta \) so that \( x^i \) is preferred to \( x^j \) if and only if \( U(Z(x^i)) > U(Z(x^j)) \). The partial derivative \( \partial U(z)/\partial z_k \) is an estimate of the marginal contribution in overall value \( \Theta \) from an additional unit of \( C_q \) in the criterion \( C_k \). That is, \( 1 \bigoplus_k = (\partial U(z)/\partial z_k) \Theta \). When \( U(Z(x)) \Theta = (\sum w_k z_k(x)) \Theta \) or \( \partial U(z)/\partial z_k \) is estimated by \( w_k \) \([2, 8, 17, 26]\), the marginal contribution is \( w_k \Theta \) per \( C_q \) as in

\[
1 \bigoplus_k = w_k \Theta
\] (1)

When \( w_k > w_q \), one unit of \( C_q \) is worth more than one unit of \( C_q \) measured in \( \Theta \). We have constant value of \( w_k = \partial U(z)/\partial z_k \) only in the linear case. When \( U \) is nonlinear, \( \partial U(z)/\partial z_k \) changes with \( z \) and iterative steps would be required. If \( \Theta \) were known explicitly, the marginal contribution \( w_k \Theta / \bigoplus_k \) can be elicited from the DM by the question “What is the overall worth of \( C_k \)?”. Such questions are not used in practice since \( \Theta \) is almost never identified explicitly in MCDM problems. In practice, the marginal contribution \( w_k \Theta / \bigoplus_k \) is rarely estimated directly. In SMART \([28]\), the criterion \( C_k \) is measured so that its worst case value is \( 0 \bigoplus_k \) and its best value \( 1 \bigoplus_k \). Then, \( w_q / w_k \) is the ratio of the contribution of the swing from worst to best in the criterion \( C_q \) to the contribution of the swing from worst to best in the criterion \( C_k \).

In conjoint analysis \([5]\), \( U(R(x)) \Theta = \sum w_k r_k(x) \Theta \) is assumed and the marginal contributions are estimated indirectly from experts’ prior holistic judgment. The criteria weights \( w_k \) are obtained by maximizing the consistency between the ranking recovered from \( \sum w_k r_k(x) \) and the prior known preference ranking or rating from experts’ holistic judgment in some form of goal programming. With \( w_k \) estimating \( \partial U/\partial r_k \), we have \( 1 \bigoplus_k = w_k \Theta \) and the alternatives may be modified and improved based on \( w_k \Theta / \bigoplus_k \), \( k = 1, \ldots, p \). This is frequently used for product design in marketing research.

In linking pin AHP \([23]\) with \( x \) being the linking alternative (for simplicity), the criteria impacts are transformed so that \( r_k(x^1) \bigoplus_k = 1 \bigoplus_k \) and \( r_k(x^i) / r_k(x^1) = r_k(x^i) \) is the relative ratio of the desirability of \( x^i (i = 2, \ldots, m) \) to \( x^1 \) in the criterion \( C_k \), \( k = 1, \ldots, p \). Then \( w_k \) is the marginal contribution of \( 1 \bigoplus_k \) to \( w_k \Theta \). The ratio \( w_q / w_k \) of criteria weights is the answer to the question “What is the ratio of the overall worth of \( 1 \bigoplus_q \) to that of \( 1 \bigoplus_k \)?”. In practice, this question is being asked in the form “Which is more important, the worth of \( x^1 \) (the linking alternative) in the criterion \( C_q \) or the worth of \( x^1 \) in the criterion \( C_k \), and by how many times?”.

In multiplicative AHP \([12]\),

\[
U(Z(x^j)) = \prod z_k(x^j)^{w_k}
\]

and \( \partial U(z^*)/\partial z_k = w_k U(z^*)/z_k^* \). Thus, \( w_k \) may not be the marginal contribution of \( 1 \bigoplus_k \). As

\[
\log \left( \prod z_k(x^j)^{w_k} \right) = \sum w_k \log(z_k(x))
\]

is in linear form, it follows that \( w_k \) can be interpreted as the marginal contribution of one unit of \( \log(z_k(x)) \) into units of \( \log[U(Z(x))] \). The overall ranking derived from using the multiplicative form and the linear form \( \sum w_k \log(z_k(x)) \) would be similar since the logarithm function \( \log \) is monotone and increasing.
4.2. Weight ratios as indifference trade-offs

In the neighborhood of \( z^* = Z(x^*) \), the contour \( \{ z: U(z) = U(z^*) \} \) is estimated by the tangent plane \( \nabla U(z^*)^T(z - z^*) = 0 \) which can also be written as \( \sum (\partial U(z^*)/\partial z_k)z_k = \sum (\partial U(z^*)/z_k)z^*_k \). To maintain \( U(z) = U(z^*) \), a decrease of \( \Delta k \) must be compensated by an increase of \( \Delta q \), with all other things being equal, where \( \Delta k \) and \( \Delta q \) must satisfy:

\[
(\partial U(z^*)/\partial z_k)(z_k - \Delta k) + (\partial U(z^*)/\partial z_q)(z_q + \Delta q) = (\partial U(z^*)/\partial z_k)z_k + (\partial U(z^*)/\partial z_q)z_q
\]

\[
\Leftrightarrow (\partial U(z^*)/\partial z_k)\Delta k = (\partial U(z^*)/\partial z_q)\Delta q
\]

\[
\Leftrightarrow \Delta_k = \Delta_q(\partial U(z^*)/\partial z_q)/(\partial U(z^*)/\partial z_k)
\]

With \( \Delta q = 1 \), the indifference trade-offs of \( 1 \) is

\[
1 \odot q = \left[ (\partial U(z^*)/\partial z_q)/(\partial U(z^*)/\partial z_k) \right] \odot k \tag{2}
\]

The trade-offs ratios (Eq. 2) represent the relative value of unit changes of the criteria. Obviously, the trade-offs ratios depend on the units \( \odot k \) and thus the magnitudes of the criteria weights do not represent the relative importance of the criteria.

When \( U(Z(x))\Theta = (\sum w_k z_k(x))\Theta \) or \( \partial U(z^*)/\partial z_k \) is estimated by \( w_k, \ k = 1, \ldots, p \) \([2, 8, 17, 26]\), the indifference trade-offs in Eq. 2 becomes

\[
1 \odot q = (w_q/w_k) \odot k \tag{3}
\]

This interpretation is used frequently in interactive compensatory methods \([26]\) to estimate the criteria weights \( w_1, \ldots, w_p \). The ratio \( w_q/w_k \) is the answer to the question “How many \( \odot k \) units are required to compensate for a loss of \( 1 \odot q \) unit?” In practice, the trade-offs \( w_q/w_k \) may be approximated by asking the DM “With all scores in other criteria held constant, how much increase in the criterion \( C_k \) are you willing to accept as compensation for one unit loss in the criterion \( C_q \)?”

4.3. Weights as gradient of MAV function

The gradient \( \nabla U(z) \) of the overall value function \( U(Z(x)) \) at \( z = Z(x) \) is \([\partial U(z)/\partial z_1, \ldots, \partial U(z)/\partial z_p]\) and changes direction with \( z = Z(x) \) when \( U \) is nonlinear. In nonlinear optimization, \( \nabla U(z) \) points to the best direction for increasing \( U(z) \). This property is naturally used in multiobjective interactive method where the gradient \( \nabla U(z) \) is estimated iteratively in search of new improved alternatives \( z + z\nabla U(z) \) for some step size \( z \). When \( U(Z(x))\Theta = (\sum w_k z_k(x))\Theta \) or the gradient \( \nabla U(z) \) is estimated by the weight vector \( (w_1, \ldots, w_p) \) \([8, 17, 26]\), the criteria weights can be interpreted and elicited as the “optimal” mix of additional units of \( z_1(x)\odot 1, \ldots, z_p(x)\odot p \). The dependence of \( w_1, \ldots, w_p \) on \( z \) has led to the idea of state dependent criteria weights \([30]\). In practice, the gradient \( \nabla U(z) \) is estimated indirectly via marginal contributions or trade-offs.
4.4. Weights as scaling factors

Whenever \( U(\mathcal{R}(x)) \) is estimated by a linear sum weighted by \( w_1, \ldots, w_p \), the criteria weight \( w_k \) is a scaling factor which converts each criterion unit into \( w_k \Theta \). For \( U(\mathcal{R}(x)) \Theta = (\sum w_k r_k(x)) \Theta \), the criterion weight \( w_k \Theta / \mathcal{R}_k \) converts \( 1 \mathcal{R}_k \) to \( w_k \Theta \). We would use \( w_k \Theta / \mathcal{R}_k \) as the proper conversion factor just as 100 cents/$ is used for $1 = 100 cents. With unknown \( \Theta \), the criteria weights \( w_k \) are not elicited from the DM directly as scaling factors. It is not practical to ask the question “What is the overall worth of 1\mathcal{R}_k?” Yet it is important to note the criteria weights \( w_k \Theta / \mathcal{R}_k \) are used as conversion factors in \( U(\mathcal{R}(x)) \Theta = (\sum w_k r_k(x)) \Theta \).

In goal programming and compromise (ideal point) programming \([2,11,26]\), the overall objective is to minimize a weighted sum of deviations \( \sum w_k d(z_k(x), t_k) \) where \( t_k \) is the ideal or targeted value in the criterion \( C_k \) and \( d(z_k(x), t_k) \) is a deviational measure of distance between \( z_k(x) \) and \( t_k \). Again \( w_k \) is a scaling factor which converts each \( \mathcal{C}_k \) distance between \( z_k \) and \( t_k \) to \( w_k \Theta \). The weights \( w_1, \ldots, w_p \) depend on the ideal point (goal targets) and the goal constraints. An extremely high target \( t_k \) would lead to smaller value of \( w_k \) such that there is less severity in its underachievement. Obviously, \( w_k \) changes with \( \mathcal{C}_k \) and \( \mathcal{R}_k \). As scaling factors, the magnitudes of such criteria weights do not necessarily measure the relative importance of the criteria.

4.5. Weights as coefficients in linear overall value function

Suppose \( U \) is linear in \( z \) and \( U(\mathcal{Z}(x)) \Theta = (\sum w_k z_k(x)) \Theta \) \([2,8,17,26]\). Then \( \partial U(z) / \partial z_k = w_k \), \( k = 1, \ldots, p \), and the interpretations of the coefficient \( w_k \) as marginal contribution in Eq. 1, as trade-offs in Eq. 3, and as a scaling factor which converts \( 1 \mathcal{C}_k \) in the criterion \( C_k \) to \( w_k \Theta \) of overall value are valid. Thus, \((1/w_q) \mathcal{C}_q = (1/w_k) \mathcal{C}_k = 1 \Theta \). It follows from the discussions above that the criteria weights \( w_1, \ldots, w_p \) are determined indirectly from the ratio \( w_q/w_k \) which is elicited from the DM by asking questions such as “With all scores in other criteria held constant, how much increase in the criterion \( C_q \) would lead to smaller value of \( w_k \) so that there is less severity in its underachievement? Obviously, \( w_k \) changes with \( \mathcal{C}_k \) and \( \mathcal{R}_k \). Thus the magnitudes of such criteria weights do not necessarily measure the relative importance of the criteria.

In AHP, the criteria weights \( w_k \) in \( U(\mathcal{R}(x)) \Theta = \sum w_k r_k(x) \Theta \) can be interpreted as scaling factor \( w_k \Theta / \mathcal{R}_k \), as marginal contribution in \( 1 \mathcal{R}_k = w_k \Theta \) and as trade-offs in \( 1 \mathcal{R}_q = (w_q/w_k) \mathcal{R}_k \). The relevant questions are “What is the ratio of the overall worth of \( 1 \mathcal{R}_q \) to that of \( 1 \mathcal{R}_k \)?” and “How many \( \mathcal{R}_k \) units are required to compensate for a loss of \( 1 \mathcal{R}_q \) unit?” These questions are difficult to answer and are not used directly in practice. In linking pin AHP with \( x^1 \) as the linking alternative, the question “Which is more important, the worth of \( x^1 \) (the linking alternative) in the criterion \( C_q \) or the worth of \( x^1 \) in the criterion \( C_k \), and by how many times?” is used to elicit the ratios \( w_q/w_k \) from the DM. In the conventional AHP, the actual question asked is “Of the two criteria \( C_q \) and \( C_k \) being compared, which one is considered more important, and by how many times, with respect to the overall goal?” which has a rather vague and ambiguous meaning \([22]\). This is further complicated by the sum to one normalization imposed on the criteria weights. The normalization \( w_1 + \cdots + w_p = 1 \) is a nontrivial constraint which imposes the restriction that a bundle consisting of \( 1 \mathcal{R}_1, \ldots, 1 \mathcal{R}_p \) is
worth $\Theta$. When a new alternative is added, the “sum to one” normalization constraint in the conventional AHP changes from the scale of $\mathbb{R}_k$ to a new scale $\mathbb{R}_k'$. Consequently, $w_k\Theta/\mathbb{R}_k$ has to be modified accordingly. Failing to replace $\Theta/\mathbb{R}_k$ by $\Theta/\mathbb{R}_k'$ may lead to incorrect evaluations and, in the extreme case, incorrect ratios (and even illegitimate rank reversals) which should be regarded as a disturbing property [21,22]. Criteria weights can not be determined correctly without proper adaptation of the problem context and the aggregation rule.

4.6. Weights as relative contribution to overall value

In AHP, $U(R(x))$ is estimated by $\sum w_k r_k(x)\Theta = w_k r_k(x)\Theta$ is the partial value of $x$ in the criterion $C_k$. The criteria weight is a scaling factor which converts $1/\mathbb{R}_k$ to $w_k\Theta$ and $r_k(x^i)/r_k(x^j)$ is the relative ratio of the desirability of $x^i$ to $x^j$ in the criterion $C_k$. Suppose there are $m$ alternatives $x_1, \ldots, x_m$ to be evaluated. In conventional AHP, it follows from $r_k(x^1) + \cdots + r_k(x^m) = 1$ that $w_k = w_k r_k(x^1) + \cdots + w_k r_k(x^m)$ is the sum of the partial values of all the alternatives in the criterion $C_k$, $k = 1, \ldots, p$. Thus, $w_k$ is the total contribution or worth of all the alternatives in the criterion $C_k$. As the criteria weights are in ratio scale, we can say that $w_1, \ldots, w_p$ are proportional to the average (and total) worth of all the alternatives in $C_1, \ldots, C_p$, respectively. This leads to the interpretation of $w_k$ in the referenced AHP as the relative contribution of average scores [22]. The ratio $w_q/w_k$ of criteria weights can then be elicited by the question “What is the ratio of the average worth of all the alternatives in the criterion $C_q$ to the average worth of all the alternatives in the criterion $C_k$?”. When there are well established standards for judging the alternatives, the performance under a criterion may be categorized into indicators (intensities) such as below average, average, good and excellent. Each alternative is characterized by the indicators under the criteria. Then the ratio $w_q/w_k$ can be elicited by the question “What is the ratio of the worth of the typical (average) alternative in the criterion $C_q$ to the worth of the typical (average) alternative in the criterion $C_k$?”. This may be used in the absolute mode AHP [20] where the question is easier since it does not involve the alternatives directly.

It is common that the larger partial value $T_k(x)\Theta$ is taken as a sign of importance of the criterion $C_k$. The relative size of the partial value $T_k(x) = w_k r_k(x)$ in $\sum w_k r_k(x)$ is crucial in the preference ratio $\sum w_k r_k(x)/\sum w_k r_k(x)$ of $x^i$ to $x^j$. Thus criteria importance depends on the magnitude of the sum of partial values $T_k(x^1) + \cdots + T_k(x^m)$ of all the alternatives. It is reasonable to define the relative importance of the criterion $C_k$ as the relative contribution of the total or average score of all the alternatives [22].

4.7. Weights for discriminating power of criteria on alternatives

The ultimate purpose in MCDM is to evaluate and measure the differences between the alternatives as whole by estimating $U(Z(x))$ or $U(R(x))$. In the context of the MCDM problem, the discriminating power of the criterion $C_k$ is its relative level of usefulness in determining the differences in overall values in $\Theta$ among the alternatives. Such discriminating power is commonly regarded as the criteria importance. Most DM would expect the performances of the best alternative in more important criteria to be closer to their maxima.
Suppose \( U(Z(x)) \) is measured in an interval scale and \( U(Z(x))\Theta = (\sum w_k z_k(x))\Theta \). If all the alternatives have similar values in the criterion \( C_k \), then the contributions \( w_k z_k(x) \) of \( x^i \) and \( w_k z_k(x^j) \) of \( x^j \) cancel each other in \( U(Z(x^i))\Theta - U(Z(x^j))\Theta \) and thus, \( C_k \) is not useful in differentiating \( x^i \) and \( x^j \). This establishes a positive relationship between the discriminating power of the criterion \( C_k \) and the range of \( \{z_k(x); x \in S\} \).

In AHP, \( U(R(x))\Theta = \sum w_k r_k(x)\Theta \) is measured in a ratio scale and the ratio \( \sum w_k r_k(x^i)/\sum w_k r_k(x^j) \) is used to compare any pair of alternatives \( x^i \) and \( x^j \). Even when all the alternatives are the same in the first criterion \( C_1 \), say, so that \( w_1 r_1(x^i) = w_1 r_1(x^j) \), significant changes in the ratio \( \sum w_k r_k(x^i)/\sum w_k r_k(x^j) \) may be caused by deleting the partial values \( w_1 r_1(x^i) \) and \( w_1 r_1(x^j) \). Thus, the partial value \( T_k(x) = w_k r_k(x) \) of \( x \) in each criterion \( C_k \) is essential in evaluating the ratio \( \sum w_k r_k(x^i)/\sum w_k r_k(x^j) \) even when there is very little deviations in \( r_k(x) \). Consequently, the criteria weights are not directly related to the discriminating power of the criteria in AHP by the partial values.

### 4.8. Weights as relative contribution of swing from worse to best

In MAVT, \( U(Z(x)) = (\sum w_k z_k(x))\Theta \) is measured in an interval scale. The criteria weights are determined directly by evaluating the impact of the swings from worst to best in each criterion [28]. With the transformed criterion values \( z_k(x) \) in the interval \([0, 1]\), the swing from worse to best in the criterion \( C_k \) is from \( 0 \in C_k \) to \( 1 \in C_k \). From \( U(Z(x))\Theta = (\sum w_k z_k(x))\Theta \), we see that the impact of the swing from \( 0 \in C_k \) to \( 1 \in C_k \) is \( w_k \Theta \). Thus, \( w_k/\hat{w}_k \) can be elicited by the question “What is the ratio of the contribution of the swing from worst to best in the criterion \( C_k \) to the contribution of the swing from worst to best in the criterion \( C_k \)?”. This implies that the criteria weights are proportional to the discriminating power of the criteria in MAVT and do reflect the criteria importance.

### 4.9. Weights as vote values in binary choices

The simplest but weakest measure on the alternatives \( x^1, \ldots, x^m \) under the criterion \( C_k \) is the ordinal ranking based on \( z_k(x^1), \ldots, z_k(x^m) \). These ordinal rankings under the \( p \) criteria can be used to choose between any two alternatives \( x^i \) and \( x^j \) [15]. Let \( C(x_i, x_j) = \{k: x^i \text{ ranks better than } x^j \text{ under the criterion } C_k \} \) and \( |C(x^i, x^j)| \) denotes the number of criteria in \( C(x^i, x^j) \). If each criterion is counted as one vote, then the \( |C(x^i, x^j)| \) is the number of votes supporting \( x^i \) in the binary choice among \( x^i \) and \( x^j \). When \( |C(x^i, x^j)| > p/2 \), the alternative \( x^i \) has majority support of the \( p \) criteria and is selected over \( x^j \) by majority consensus.

When the criteria have different impacts, \( w_k \Theta \) is defined to be the vote value of the criterion \( C_k \). Let \( S_{ij} \) be the sum of \( w_k \) over all \( k \) in \( C(x^i, x^j) \). Then \( S_{ij} \Theta \) is the total vote value which supports \( x^i \) in the binary choice among \( x^i \) and \( x^j \). Naturally, \( x^i \) is selected over \( x^j \) when \( S_{ij} > S_{ji} \) and \( x^j \) has more vote value than \( x^i \). The obvious weakness is that the whole \( w_k \) is added to \( S_{ij} \) even when \( x^i \) only marginally preferred to \( x^j \) under the criterion \( C_k \). The set \( C(x^i, x^j) \) is called the concordant coalition and \( S_{ij} \Theta \) is called the concordance index for supporting \( x^i \) in the binary choice among \( x^i \) and \( x^j \) [26]. The criteria weights \( w_1, \ldots, w_p \) are intrinsic vote values, which reflect the relative importance of the criteria, in the votes of binary choices. Note that only the weak ordinal information is captured in the vote value \( w_k\Theta \) since either none or the
whole $w_k \Theta$ is added to $S_q \Theta$ without any fractional modification to reflect and differentiate the strength of preference $z_k(x^i) - z_k(x^j)$ of $x^i$ to $x^j$ under the criterion $C_k$ in $C(x^i, x^j)$.

In the ELECTRE outranking method [19], $x^i$ is preferred to $x^j$ when $z_k(x^i) - z_k(x^j) \geq p_k$ for some preference threshold $p_k > 0$. The ordinal ranking of the criteria weights $w_1, \ldots, w_p$ is elicited from the DM by question such as “Is $z_q(x) + p_q$ of greater, minor or the same importance compared with $z_k(x) + p_k$ with all scores in other criteria held constant?” If the effect of $z_k(x) + p_q$ and $z_k(x) + p_k$ is of minor importance compared with $z_k(x) + p_k$, then $w_q + w_k \leq w_k$. Most DM would expect the overall ranking of the alternatives to be the same as the ranking of the alternatives under very important criteria. The total vote value $S_q \Theta$ is simply the result of counting and adding $w_k \Theta$ over the criteria in $C(x^i, x^j)$, and it only provides weak ordering on the alternatives. Thus this approach should not be used when the alternatives $x^1, \ldots, x^m$ are measured in interval or ratio scale under the criteria.

4.10. Contribution of the criteria at the optimal alternative

Most optimization techniques are based on the conditions prevailing at optimal solution $x^*$ and the larger partial value $T_k(x^*)$ at the optimal point $x^*$ may be mistaken as a sign of importance of the criterion $C_k$. When $U(R(x)) \Theta = \sum w_k r_k(x) \Theta$, the partial value is $T_k(x) \Theta = w_k r_k(x) \Theta$ which represents the portion of $U(R(x)) \Theta$ contributed by $r_k(x) \Theta$. However, the criteria weights $w_1, \ldots, w_p$ are in general not proportional to $T_1(x^*), \ldots, T_p(x^*)$. For general nonlinear $U(R(x))$, it is well known in optimization theory that some optimal solution may not be obtained by using the weighted sum approach [21]. On the other hand, it often happens that different criteria weights give the same efficient solution in linear multiple objective programming [17]. Thus, the achievement levels at the optimal alternative $x^*$ are not necessarily related to the criteria weights nor the criteria importance.

4.11. Weights as parameters in optimization

In optimizing $U(Z(x))$, the criteria weight vector $(w_1, \ldots, w_p)$ is frequently used to estimate the gradient $\nabla U(z)$ [2,8,17,26] which points to the best direction for increasing $U(z)$. The criteria weights can be regarded as parameters which are used to search for better solutions. In the interactive goal programming and Tchebycheff methods [25], the criteria weights are parameters in the weighted sum of deviations $\sum \omega_t d(z_k(x), t_k)$, where $t_k$ is the ideal or targeted value in the criterion $C_k$ and $d(z_k(x), t_k)$ is a deviational measure of distance between $z_k(x)$ and $t_k$. Again, the criteria weights are perturbed to search for better solutions. The local interpretation of the criteria weights as $1 \Theta_q = (w_q/w_k) \Theta$ trade-offs may improve the searching directions. Proper convexity conditions are necessary for the existence of an optimal alternative to be obtained by some criteria weights [17]. As parameters in the searching process, the criteria weights are changing with $z$ and do not reflect the criteria importance.

4.12. Weights for information content carried in criteria

Suppose $U(Z(x))$ is measured in an interval scale and $U(Z(x)) \Theta = (\sum w_k z_k(x)) \Theta$. The contrast intensity of the criterion $C_k$ is the information content of $z_k(x^1), \ldots, z_k(x^m)$ which is
positively related to their range and standard deviation \( \sigma_q \). The higher the contrast intensity of \( C_q \), the more information content in \( C_q \) available to be used. For the criterion \( C_q \) to be more important than the criterion \( C_k \), the values \( z_q(x^1), \ldots, z_q(x^m) \) cannot be similar to the values \( z_k(x^1), \ldots, z_k(x^m) \). The conflicting character of \( C_q \) is negatively related to the correlation coefficient \( \rho_{qk} \) between \([z_q(x^1), \ldots, z_q(x^m)]\) and \([z_k(x^1), \ldots, z_k(x^m)]\), \( k = 1, \ldots, p \) and \( k \neq q \). Thus, a necessary condition for \( C_q \) to have high information content is high contrast intensity with large \( \sigma_q \) and high conflict character with large \( \rho_{qk} \). In the CRITIC method [6], the criterion weights are computed by normalizing \( \sigma_q \sum_k (1 - \rho_{qk}), \quad q = 1, \ldots, p \). However, there is no guarantee that the magnitude of \( w_q \) would reflect the criteria importance of \( C_q \) because it is possible that the criterion \( C_q \) is least important and yet \( z_q(x^1), \ldots, z_q(x^m) \) have very high information content which is not useful for identifying the best alternative.

4.13. Weights for relative functional importance of criteria

In many MCDM models, the criteria weights are elicited from the DM as the relative importance of the criteria without defining the word “importance” clearly. This has resulted in confusion and misunderstanding in using MCDM models [21,22]. It is evident that criteria importance must be closely related to the purpose of some specific functions which are crucial in evaluating the overall desirability of the alternatives. The relative importance of the criterion \( C_q \) is related to the extent that the rankings of the alternatives under \( C_q \) are the same as their overall ranking [16,18]. Thus the importance of \( C_k \) in ranking the alternatives is given by the Spearman rank correlation between \([U(Z(x^1)), \ldots, U(Z(x^m))]\) and \([z_k(x^1), \ldots, z_k(x^m)]\).

We submit that the important functions of criteria weights in MCDM are: (a) the power of discriminating and differentiating the overall desirability of the alternatives, and (b) the evaluation of the preference ratio \( \sum w_k r_k(x^i) / \sum w_k r_k(x^j) \) of \( x^i \) to \( x^j \). For the magnitudes of \( w_1, \ldots, w_p \) to represent criteria importance, \( w_1, \ldots, w_p \) must be independent of \( \tilde{C}_1, \ldots, \tilde{C}_p \) which can be rescaled and normalized.

When criteria importance is defined as the power of the criteria in discriminating the overall desirability of the alternatives, the criteria swing weights do represent the relative importance of the criteria as in MAVT [28]. When criteria importance is defined as the impact on the preference ratio \( \sum w_k r_k(x^i) / \sum w_k r_k(x^j) \), the criteria weights in referenced AHP [22] do represent the relative importance of the criteria. Criteria importance may also be defined as the partial values of a typical or representative alternative \( x^l \). Such criteria importance can be captured in the linking pin AHP [23] with \( x^l \) as the linking alternative. To avoid unnecessary confusion, the generic term “criteria importance” should be explicitly defined whenever it is used in any MCDM model.

5. Summary and conclusion

Criteria weights have been used widely with a long list of plausible interpretations. Table 1 gives the summary of the plausible interpretations of criteria weights. In most cases, the criteria weights are used to represent marginal contributions, criteria trade-offs, scaling factors and criteria importance. Interpretation as marginal contributions or trade-offs is not operationally
effective due to the dependence on the measurement units which may not be known explicitly. In MAVT, the overall values are in some interval scale and the criteria weights represent the criteria importance in discriminating power which is proportional to the swing from worst to best in each criterion. In reference AHP [22], the overall values are in a ratio scale and the criteria weights represent the criteria importance which is proportional to the total score of the alternatives in each criterion. In linking pin AHP [23], the criteria weights are proportional to the marginal worth of the linking alternative in each criterion.

Proper interpretation of criteria weights is influenced by the manner they are used in the aggregation rule of the MCDM model. Based on the commensurability and scale validity, we determine the appropriate questions posed to the DM for eliciting information on criteria.

Table 1
Meaning of criteria weights and MCDM models

<table>
<thead>
<tr>
<th>Models interpretation</th>
<th>Linking pin AHP</th>
<th>Referenced AHP</th>
<th>Saaty’s AHP</th>
<th>Interactive MCDM</th>
<th>MVT SMART</th>
<th>Outranking ELECTRE</th>
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<tr>
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<td>Yes</td>
<td>Yes</td>
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<td>Yes</td>
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<td>Yes</td>
<td>Yes</td>
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<td>No</td>
</tr>
<tr>
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<td>Yes</td>
<td>Yes</td>
<td>No</td>
<td>Yes</td>
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</tr>
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</table>

Question to DM

a Which is more important, the worth of $x^1$ (the linking alternative) in the criterion $C_q$ or the worth of $x^1$ in the criterion $C^*$, and by how many times?
b Which is more important, the average worth of all the alternatives in the criterion $C_q$ or the average worth of all the alternatives in the criterion $C_k$, and by how many times?
c Of the two criteria $C_q$ and $C_k$ being compared, which one is considered more important, and by how many times, with respect to the overall goal?
d With all scores in other criteria held constant, how much increase in the criterion $C_q$ are you willing to accept as compensation for one unit loss in the criterion $C_k$?
e How much more important is criterion $C_q$ than the least important criterion?
f What is the ratio of the contribution (to overall value) of the swing from worst to best in the criterion $C_q$ to the contribution of the swing from worst to best in the criterion $C_k$?
g With all scores in other criteria held constant, is the increase in the criterion $C_q$ by its preference threshold $p_{q^*}$ of greater, minor or the same importance compared with the increase in the criterion $C_k$ by its preference threshold $p_{k^*}$?
weights and the appropriate aggregation rule for each possible interpretation of criteria weights. The generic term “criteria importance” should be explicitly defined whenever it is used in any MCDM model. We hope that proper interpretation and applications of criteria weights would enhance the quality of the results obtained by using the multitude of MCDM models.

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References